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Existence of smooth attractors for the Navier–Stokes-omega model of turbulence

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ABSTRACT

We consider the Navier–Stokes-$\overline{\omega}$ model, given by

$$u_t - u \times \overline{\omega} - \nu \Delta u + \nabla P = f, \qquad \omega = \nabla \times u, \qquad \nabla \cdot u = 0$$

subject to periodic boundary conditions with zero mean. The NS-$\overline{\omega}$ model is an outgrowth of ideas in approximate deconvolution models and in NS-alpha models. Like the NS-alpha model, it is simple and conserves, in the appropriate context, kinetic energy and helicity (3d) or energy and enstrophy (2d). In first tests NS-$\overline{\omega}$ was found to be accurate, robust and amenable to efficient numerical simulation. In this note we prove existence and regularity of a global attractor for the model.

1. Introduction

The Navier–Stokes-$\overline{\omega}$ model [21,18] is a development from the NS-alpha model circle of ideas, e.g., [9], and approximate deconvolution large eddy simulation models, e.g., [32,33,1,13,15,7,29,19,28]. In rotational form, it is to find a velocity u and Bernoulli or dynamic pressure P satisfying

$$u_t - u \times \overline{\omega} - \nu \Delta u + \nabla P = f, \qquad \omega = \nabla \times u, \qquad \nabla \cdot u = 0. \tag{1.1}$$

In (1.1) $\overline{\omega} = \nabla \times \overline{u}$ denotes the filtered averaged/smoothed vorticity, where the filter is defined precisely in Section 2 in (2.1). We consider (1.1) in $\Omega = (0, 2\pi)^3$ subject to periodic with zero mean boundary conditions on u, P, f and the initial condition $u(x, 0) = u_0(x)$. Here u_0, f are smooth, zero mean periodic functions. Minimally we suppose

$$u_0, f \in L_0^2(\Omega), \quad \nabla \cdot u_0 = \nabla \cdot f = 0, \quad \text{and} \quad \int_\Omega f \, dx = \int_\Omega u_0 \, dx = 0.$$

The NS-$\overline{\omega}$ model (1.1) (nonlinearity $-u \times \nabla \times \overline{u}$) is a basic regularization of the NSE similar in spirit to the Leray regularization (nonlinearity $+\overline{u} \cdot \nabla u$ [23,24,3]) and the NS-alpha model (nonlinearity $-\overline{u} \times \nabla \times u$ [8]). It also can be extended to a family of NS-$\overline{\omega}$-deconvolution models of arbitrary high accuracy. The idea of using deconvolution operators to obtain high order accurate regularizations is an idea of A. Dunca (private communication) developed for the Leray model in [16,17], the NS-α model in [29] and the NS-$\overline{\omega}$ model in [17]. The NS-$\overline{\omega}$ deconvolution family, including (1.1) as the zeroth order case, is given by

$$u_t - u \times D(\overline{\omega}) - \nu \Delta u + \nabla P = f, \qquad \omega = \nabla \times u, \qquad \nabla \cdot u = 0, \tag{1.2}$$

where $D : L^2(\Omega) \to L^2(\Omega)$ is a deconvolution operator.

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In this report we prove existence of an attractor for (1.1), (1.2). Let $\mathbf{H}$ denote the closure of the smooth, periodic, zero mean, divergence free, vector functions in $L^2(\Omega)$ and $\mathbf{H}^s_\#(\Omega)$ their closure in the H^s norm; see Section 2.

Theorem 1.1 *(Existence of an attractor). Let averaging be defined by the differential filter $\overline{v} = (-\alpha^2\Delta + 1)^{-1} v$. Suppose D is a bounded, linear deconvolution operator that is smoothing in the sense that*

$$\left\|D(\overline{v})\right\|_{\mathbf{H}^2_\#} \leqslant C\|v\|. \tag{1.3}$$

Suppose

$$u_0, f \in \mathbf{H}^1_\#(\Omega), \quad \nabla \cdot u_0 = \nabla \cdot f = 0, \qquad \int_\Omega f\,dx = \int_\Omega u_0\,dx = 0.$$

Then the NS-$\overline{\omega}$ deconvolution model (1.2)*, including the NS-$\overline{\omega}$ model* (1.1) *when $D = I$, has a maximal global attractor in* $\mathbf{H}$*.*

In Section 4 we show that the maximal attractor is also an attractor in each $\mathbf{H}^s_\#(\Omega)$ and thus consists of $\mathbf{C}^\infty_\#(\Omega)$ functions. This parallels known results for the 2d NSE, Temam's Chapter IV, Section 6.3 in [34]; in the latter case, it is proven through establishing regularity of time derivatives and herein through space derivatives. Some preliminaries and the (standard) notation used are collected in Section 2. The proof of the above theorem is given in Section 3. The theory of attractors is highly developed, e.g., Temam [34], Robinson [30] and applied to the Leray and Leray deconvolution model in Cheskidov, Holm, Olson and Titi [3], and by Lewandowski and Preaux [25] (the last report inspired this effort). Under this theory, the proof of the above theorem reduces to verifying (i) existence of a bounded, absorbing set in $\mathbf{H}$ and (ii) compactness of the semigroup generated by (1.1), (1.2). The question of the dimension of the attractor and its dependence on the averaging radius α and deconvolution operator D (and how it compares with estimates of length scales of persistent eddies from turbulence phenomenology as well as estimates of attractor dimension of the Leray and NS-alpha regularizations) is a very important open problem.

2. Notation and preliminaries

$(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual $L^2(\Omega)$ inner product and norm. The subscript # denotes the 2π periodic so the $C^\infty_\#(\Omega)$ denotes the C^∞ functions v with v and all derivatives 2π periodic. Let

$$\mathbf{H}_{\mathrm{div}}(\Omega) := \text{closure in } \|\cdot\|_{\mathrm{div}} := \sqrt{\|\cdot\|^2 + \|\operatorname{div}\cdot\|^2} \text{ of } \left\{v \in C^\infty_\#(\Omega)^3\colon \int_\Omega v\,dx = 0\right\},$$

$$\mathbf{H}^s_\#(\Omega) := \text{closure in } \left\|\nabla^s\cdot\right\| \text{ of } \left\{v \in C^\infty_\#(\Omega)^3\colon \int_\Omega v\,dx = 0 \text{ and } \nabla\cdot v = 0\right\}.$$

The norm on $\mathbf{H}^s_\#(\Omega)$ can be defined succinctly via Fourier series as

$$\|v\|_s^2 = \sum_{\mathbf{k}} |\mathbf{k}|^{2s}\left|\widehat{v}(\mathbf{k})\right|^2.$$

Define

$$\mathbf{H} = \{v \in \mathbf{H}_{\mathrm{div}}\colon \nabla\cdot v = 0\}, \quad \text{and} \quad \mathbf{V} := \mathbf{H}^1_\#(\Omega).$$

Let P_L denote the Helmholtz–Leray orthogonal projection of $L^2(\Omega)$ onto $\mathbf{H}$. The Stokes operator A is then $A = -P_L\Delta$ with $Domain(A) = \mathbf{H} \cap \mathbf{H}^2_\#(\Omega)$, e.g., Galdi [11]. In the periodic case the vector Laplacian and Stokes operator coincide apart from their domains of definition. Let

$$\lambda_1 = \lambda_{\min}(A) = \min_{v\in\mathbf{V}} \frac{(\nabla v, \nabla v)}{\|v\|^2} \quad \text{so that} \quad (f, v) \leqslant \frac{1}{\lambda_1}\|f\|\|\nabla v\|.$$

2.1. Filtering and deconvolution

Koenderink [14], see also [26], has proven that the only filter satisfying the basic requirements of linearity, frame invariance and scale invariance is the Gaussian filter. Calculations with Gaussian filters can be expensive so *approximations* to the Gaussian are often used, such as the Pao filter, Pope [27], Sagaut [31],

$$\overline{v}(x) := \sum_{\mathbf{k}} \left(\alpha^2|\mathbf{k}|^2 + 1\right)^{-1}\widehat{v}(\mathbf{k})e^{i\mathbf{k}\cdot x}, \quad \alpha := \text{filter radius}.$$

The Pao filter is equivalent to a simple differential filter, Germano [12], used in theoretical studies of deconvolution models, e.g., [7,6,1,15], and the NS-alpha model [8]. The equivalent differential filter to the Pao filter is

$$\bar{v} = \left(-\alpha^2\Delta + 1\right)^{-1} v, \tag{2.1}$$

subject to periodic boundary conditions. In the periodic case, this differential filter preserves zero mean and incompressibility.

A deconvolution operator D is an *approximate* filter inverse. Typically D is a high order approximate inverse on the low Fourier modes and includes some sort of truncation or regularization to suppress the growth of noise in higher modes. Since many examples exist (due to approximate deconvolution's centrality in image processing, Bertero and Boccacci [2]) we assume some basic properties of D common to many used in practice:

- stability: $D : \mathbf{H} \to \mathbf{H}$ is a *bounded* linear operator,
- accuracy: $\|v - D(\bar{v})\| \leqslant C(v)\alpha^{\beta}$, for smooth v and some $\beta > 2$,
- smoothing: $\|D(\bar{v})\|_2 + \|D(\bar{v})\|_1 + \|D(\bar{v})\| \leqslant C(\alpha)\|v\|$, for all $v \in \mathbf{H}$.

Examples of filters satisfying these minimal conditions include van Cittert deconvolution [2], $D = \sum_{j=0}^{J}(-\alpha^2\Delta + 1)^{-j}$ (and its optimized variants [20,22]), Tikhonov regularized deconvolution, $D = [(-\alpha^2\Delta + 1)^{-1} + \mu I]^{-1}$, and truncated SVD methods (which simplify in the periodic case) such as

$$Dv := \sum_{0<|\mathbf{k}|<\pi/\alpha} \left(\alpha^2|\mathbf{k}|^2 + 1\right)\widehat{v}(\mathbf{k})e^{i\mathbf{k}\cdot x} + \sum_{|\mathbf{k}|\geqslant\pi/\alpha} \widehat{v}(\mathbf{k})e^{i\mathbf{k}\cdot x}.$$

2.2. A priori bounds and two Gronwall-type lemmas

The uniform Gronwall lemma of Foias and Prodi [10] is a fundamental tool in the study of attractors and used to complete the proof of existence of an attractor from (3.11) below.

Lemma 2.1 *(Uniform Gronwall lemma). Assume that y, g, h are positive, locally integrable functions on (t_0, ∞), and that for $t \geqslant t_0$*

$$y' \leqslant gy + h, \quad \text{with}$$
$$\int_t^{t+r} y(s)\,ds \leqslant k_1, \qquad \int_t^{t+r} g(s)\,ds \leqslant k_2, \qquad \int_t^{t+r} h(s)\,ds \leqslant k_3,$$

where k_1, k_2, k_3, and r are four positive constants. Then,

$$y(t + r) \leqslant \left(\frac{k_1}{r} + k_3\right)e^{k_2}, \quad \text{for all } t \geqslant t_0. \tag{2.2}$$

Because the differential inequality (3.10) has sublinear growth in the nonlinear term, other related bounds can be obtained using an alternate Gronwall-type inequality.

Lemma 2.2. *Let $y(t)$ be a positive smooth function satisfying*

$$y' + \nu\lambda_1 y \leqslant A + By^{\frac{3}{4}}, \quad \text{for } t > 0, \qquad y(0) = y_0,$$

where A, B, y_0, ν, λ_1 are positive constants. Let $t_0 := \max\{\frac{2}{\nu\lambda_1}\ln(\frac{\nu\lambda_1 y_0}{2A}), 0\}$. Then,

$$y(t) \leqslant \max\left\{y_0, \left(\frac{2B}{\nu\lambda_1}\right)^4, \frac{4A}{\nu\lambda_1}\right\}, \quad \text{for } t \geqslant 0,$$
$$y(t) \leqslant \max\left\{\left(\frac{2B}{\nu\lambda_1}\right)^4, \frac{4A}{\nu\lambda_1}\right\}, \quad \text{for } t \geqslant t_0.$$

Proof. Define $Y := \max\{(\frac{2B}{\nu\lambda_1})^4, \frac{4A}{\nu\lambda_1}\}$ and divide $(0, \infty)$ into two collections of subintervals by

$$I_S := \left\{t:\ y(t) < Y\right\} \quad \text{and} \quad I_L := \left\{t:\ y(t) \geqslant Y\right\},$$

so that $y(t) < Y$ for all $t \in I_S$. Thus, consider $t \in I_L$. Let $[a, b]$ be a maximal interval in I_L so either $a = 0$ and $y(a) = y_0$ or $a > 0$ and $y(a) = Y$. By construction, on I_L

$$\frac{1}{2}\nu\lambda_1 y(t) \geqslant By(t)^{\frac{3}{4}},$$

so that for $t \in [a, b]$, $a > 0$, $y(t)$ satisfies

$$y' + \frac{1}{2}\nu\lambda_1 y \leqslant A, \quad a \leqslant t \leqslant b, \qquad y(a) = Y.$$

An integrating factor then gives for $t \in [a, b]$,

$$y(t) \leqslant e^{-\frac{1}{2}\nu\lambda_1(t-a)}Y + \frac{2A}{\nu\lambda_1}\left[1 - e^{-\frac{1}{2}\nu\lambda_1(t-a)}\right] \leqslant e^{-\frac{1}{2}\nu\lambda_1(t-a)}Y + Y\left[1 - e^{-\frac{1}{2}\nu\lambda_1(t-a)}\right] \leqslant Y.$$

There remains the case $a = 0$, i.e., $[a, b] = [0, b]$. The same analysis as the last case gives the bound

$$\begin{aligned} y(t) &\leqslant e^{-\frac{1}{2}\nu\lambda_1(t-a)}y_0 + \frac{2A}{\nu\lambda_1}\left[1 - e^{-\frac{1}{2}\nu\lambda_1(t-a)}\right] \\ &\leqslant e^{-\frac{1}{2}\nu\lambda_1(t-a)}\max\{y_0, Y\} + \max\{y_0, Y\}\left[1 - e^{-\frac{1}{2}\nu\lambda_1(t-a)}\right], \quad \text{or} \\ y(t) &\leqslant \max\{y_0, Y\}, \end{aligned} \tag{2.3}$$

completing the proof of the first bound. For the second, by a direct calculation we have

$$e^{-\frac{\nu\lambda_1}{2}t}y_0 \leqslant \frac{2A}{\nu\lambda_1}, \quad \text{for } t \geqslant t_0.$$

Thus, from (2.3) for $t \geqslant t_0$, the second follows since

$$y(t) \leqslant \frac{2A}{\nu\lambda_1} + \frac{2A}{\nu\lambda_1}\left[1 - e^{-\frac{1}{2}\nu\lambda_1 t}\right] \leqslant \frac{4A}{\nu\lambda_1}. \qquad \square$$

3. Existence of an attractor

The following was proven about the NS-$\overline{\omega}$ model in [21]. The same proof can be used to prove existence for the NS-$\overline{\omega}$ deconvolution model with van Cittert or Tikhonov deconvolution.

Theorem 3.1 *(Existence, uniqueness and regularity). Let $\alpha > 0$ and $T > 0$ be fixed. Let the filter be the differential filter* (2.1)*. For $u_0 \in \mathbf{V}$, $f \in \mathbf{H}$, there exists a unique strong solution u to the NS-$\overline{\omega}$ model* (1.1) *with*

$$u \in L^\infty(0, T; \mathbf{V}) \cap L^2\big(0, T; \mathbf{H}^2_\#(\Omega)\big) \quad \text{and} \quad u_t \in L^2\big((0, T) \times \Omega\big).$$

Further, u satisfies the energy equality. If the data is more regular

$$u_0 \in \mathbf{V} \cap \mathbf{H}^{m+1}_\#(\Omega), \quad \text{and} \quad f \in L^2\big(0, T; \mathbf{H}^m_\#(\Omega)\big),$$

then

$$u \in L^\infty\big(0, T; \mathbf{H}^{m+1}_\#(\Omega)\big) \cap L^2\big(0, T; \mathbf{H}^{m+2}_\#(\Omega)\big), \qquad P \in L^2\big(0, T; \mathbf{H}^{m+2}_\#(\Omega)\big).$$

The NS-$\overline{\omega}$ model is thus a well-defined dynamical system and determines a (nonlinear) semigroup defined by

$$u(t; \cdot) := S(t)u_0.$$

Proposition 3.2. *Suppose $\|\nabla\bar{u}\| + \|\Delta\bar{u}\| \leqslant C(\alpha)\|u\|$ and let*

$$t_1 := \max\left\{\frac{2}{\nu\lambda_1}\ln\left(\frac{\nu^2\lambda_1\|\nabla u_0\|^2}{4\|f\|^2}\right), 0\right\}.$$

Then, solutions to the NS-$\overline{\omega}$ model satisfy the a priori bounds

$$\big\|u(t)\big\|^2 \leqslant \|u_0\|^2 e^{-\nu\lambda_1 t} + \left(\frac{1}{\nu\lambda_1}\|f\|\right)^2\big(1 - e^{-\nu\lambda_1 t}\big), \quad \textit{for } t \geqslant 0, \tag{3.1}$$

$$\big\|\nabla u(t)\big\|^2 \leqslant \max\left\{\|\nabla u_0\|^2, \frac{16C(\alpha)^4(\sup_{[0,\infty)}\|u\|)^{10}}{\nu^8\lambda_1^4}, \frac{8\|f\|^2}{\nu^2\lambda_1}\right\}, \quad \textit{for } t \geqslant 0, \tag{3.2}$$

$$\big\|\nabla u(t)\big\|^2 \leqslant \max\left\{\frac{16C(\alpha)^4(\sup_{[0,\infty)}\|u\|)^{10}}{\nu^8\lambda_1^4}, \frac{8\|f\|^2}{\nu^2\lambda_1}\right\}, \quad \textit{for } t \geqslant t_1. \tag{3.3}$$

Proof. Since the NS-$\overline{\omega}$ model has a unique strong solution, we may multiply (1.1) by u and ∇u and integrate over Ω. The first choice yields

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\nu\|\nabla u\|^2=(f,u). \tag{3.4}$$

Since $\|u\|^2 \leqslant \lambda_1\|\nabla u\|^2$, we have

$$\frac{d}{dt}\|u\|^2+\nu\lambda_1\|u\|^2 \leqslant \frac{1}{\nu\lambda_1}\|f\|^2$$

and thus the first bound follows:

$$\|u(t)\|^2 \leqslant \|u_0\|^2 e^{-\nu\lambda_1 t}+\left(\frac{1}{\nu\lambda_1}\|f\|\right)^2\left(1-e^{-\nu\lambda_1 t}\right). \tag{3.5}$$

For the second and third, take the inner product of (1.1) with $-\Delta u$. This gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\int_\Omega u\times\nabla\times\bar{u}\cdot(-\Delta u)\,dx+\nu\|\nabla u\|^2=(f,-\Delta u)\leqslant\|f\|\|\Delta u\|.$$

For the nonlinear term we use the following bound (which follows via Holder's inequality and the Sobolev embedding theorem as in the normal bounds on the NSE nonlinearity, e.g., Constantine and Foias [4], Temam [34])

$$\left|\int_\Omega u\times\nabla\times\bar{u}\cdot(-\Delta u)\,dx\right|\leqslant C\|u\|^{\frac{1}{4}}\|\nabla u\|^{\frac{3}{4}}\|\nabla\bar{u}\|^{\frac{1}{4}}\|\Delta\bar{u}\|^{\frac{3}{4}}\|\Delta u\|.$$

Since the filter is smoothing $\|\nabla\bar{u}\|+\|\Delta\bar{u}\|\leqslant C(\alpha)\|u\|$ so

$$\left|\int_\Omega u\times\nabla\times\bar{u}\cdot(-\Delta u)\,dx\right|\leqslant C\|u\|^{\frac{5}{4}}\|\nabla u\|^{\frac{3}{4}}\|\Delta u\|\leqslant\frac{\nu}{2}\|\Delta u\|^2+C(\alpha)\nu^{-1}\|u\|^{\frac{5}{2}}\|\nabla u\|^{\frac{3}{2}}.$$

Thus,

$$\frac{d}{dt}\|\nabla u\|^2+\nu\|\Delta u\|^2\leqslant\frac{2}{\nu}\|f\|^2+C(\alpha)\nu^{-1}\|u\|^{\frac{5}{2}}\|\nabla u\|^{\frac{3}{2}},\quad\text{or} \tag{3.6}$$

$$\frac{d}{dt}\|\nabla u\|^2+\nu\lambda_1\|\nabla u\|^2\leqslant\frac{2}{\nu}\|f\|^2+C(\alpha)\nu^{-1}\|u\|^{\frac{5}{2}}\|\nabla u\|^{\frac{3}{2}}. \tag{3.7}$$

Lemma 2.2 with $y_0=\|\nabla u_0\|^2$, $A=\frac{2}{\nu}\|f\|^2$ and $B=C(\alpha)\nu^{-1}\sup_{[0,\infty)}\|u\|^{\frac{5}{2}}$ gives

$$y(t)\leqslant\max\left\{\|\nabla u_0\|^2,\frac{16C(\alpha)^4(\sup_{[0,\infty)}\|u\|)^{10}}{\nu^8\lambda_1^4},\frac{8\|f\|^2}{\nu^2\lambda_1}\right\},\quad t\geqslant 0,$$

and for $t\geqslant t_1:=\max\{\frac{2}{\nu\lambda_1}\ln(\frac{\nu^2\lambda_1\|\nabla u_0\|^2}{4\|f\|^2}),0\}$,

$$y(t)\leqslant\max\left\{\frac{16C(\alpha)^4(\sup_{[0,\infty)}\|u\|)^{10}}{\nu^8\lambda_1^4},\frac{8\|f\|^2}{\nu^2\lambda_1}\right\}. \qquad \square$$

3.1. Basic properties of attractors

To prepare for the proof of Theorem 1.1 we collect some information about attractors from, for example, Temam [34], Robinson [30], Doering and Gibbon [5], Ladyzhenskaya [35].

Definition 3.3. We say $\mathcal{A}\subset\mathbf{H}$ is a global or maximal attractor in $\mathbf{H}$ for the dynamical system (1.1) if and only if

(i) $\mathcal{A}$ is compact in $\mathbf{H}$,
(ii) for all $t>0$, $S(t)\mathcal{A}\subset\mathcal{A}$, and
(iii) for every bounded set $B\subset\mathbf{H}$, $\rho(S(t)B,\mathcal{A}):=\sup_{v\in B}\inf_{u\in\mathcal{A}}\|u-v\|$ goes to zero as $t\to\infty$.

Definition 3.4. The set $A\subset\mathbf{H}$ is an absorbing set if and only if, for every bounded subset $B\subset\mathbf{H}$ there exists $t_1>0$ such that for all $t\geqslant t_1$ one has $S(t)(B)\subset A$.

The semigroup $S(t)$ is uniformly compact if and only if for every bounded subset $B\subset\mathbf{H}$, there exists $t_2=t_2(B)$ such that $\overline{\bigcup_{t\geqslant t_2}S(t)(B)}$ is compact.

Let $\varpi(A)$ denote the set $\varpi(A):=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}S(t)(B)}$.

The proof of the following can be found in Temam [34] and Robinson [30].

Theorem 3.5. *Suppose that there exists an absorbing bounded set A and that the semigroup $S(t)$ is uniformly compact. Then, $\mathcal{A} = \varpi(A)$ is the global attractor for the dynamical system defined by $S(t)$.*

3.2. Proof of Theorem 1.1

We shall show that $S(t)$ has an absorbing set and is compact. First we establish existence of an absorbing set. (This step follows the NSE case closely.) From Proposition 3.2, the estimate (3.1) gives

$$\left\|u(t)\right\|^2 \leqslant \|u_0\|^2 e^{-\nu\lambda_1 t} + \left(\frac{1}{\nu\lambda_1}\|f\|\right)^2 \left(1 - e^{-\nu\lambda_1 t}\right). \tag{3.8}$$

As for the NSE (e.g., Temam [34], Robinson [30]), let $\rho_0 = \frac{1}{\nu\lambda_1}\|f\|$ and $\rho' > \rho_0$. Then, from (3.1), given u_0 in $\mathbf{H}$ for $t > T := \max\{\frac{1}{\nu\lambda_1}\ln(\frac{\|u_0\|^2}{\rho'^2-\rho_0^2}), 0\}$, it follows that $\|u(t)\| \leqslant \rho'$. In other words, $B_{\rho'}(0) = \{v \in \mathbf{H}\colon \|v\| < \rho'\}$ is an absorbing set in $\mathbf{H}$.

For the second step we show $S(t)$ is uniformly compact by obtaining a uniform estimate on the $\mathbf{V}$ norm of solutions. To prove compactness the second and third bounds in Proposition 3.2 do not suffice because they require $u_0 \in \mathbf{V}$ instead of $\mathbf{H}$. Thus we shall apply the uniform Gronwall lemma. Integrating (3.4) over $(t, t+r)$ and using standard inequalities gives

$$\frac{1}{2}\left(\left\|u(t+r)\right\|^2 - \left\|u(t)\right\|^2\right) + \frac{1}{2}\int_t^{t+r} \nu\left\|\nabla u(t')\right\|^2 dt' \leqslant \frac{r}{2\nu\lambda_1}\|f\|^2.$$

For $t > T$, $u(t) \in B_{\rho'}(0)$ so $\|u(t)\| < \rho'$ and $\|u(t+r)\| < \rho'$. Thus for $t > T$

$$\int_t^{t+r} \left\|\nabla u(t')\right\|^2 dt' \leqslant \frac{r}{\nu^2\lambda_1}\|f\|^2 + \frac{2\rho'^2}{\nu}. \tag{3.9}$$

We begin with (3.6), (3.7) from the proof of Proposition 3.2:

$$\frac{d}{dt}\|\nabla u\|^2 + \nu\|\Delta u\|^2 \leqslant \frac{2}{\nu}\|f\|^2 + C(\alpha)\nu^{-1}\|u\|^{\frac{5}{2}}\|\nabla u\|^{\frac{3}{2}}. \tag{3.10}$$

The Poincaré–Friedrichs inequality can be used for the last term on the RHS to give

$$\frac{d}{dt}\|\nabla u\|^2 + \nu\|\nabla u\|^2 \leqslant \frac{2}{\nu}\|f\|^2 + C(\alpha)\nu^{-1}\|u\|^2\|\nabla u\|^2. \tag{3.11}$$

The proof can now be completed by applying the uniform Gronwall lemma. With the following identifications of y, g and h

$$y(t) = \|\nabla u\|^2, \qquad g(t) = C(\alpha)\nu^{-1}\|u\|^2, \qquad h = \frac{2}{\nu}\|f\|^2,$$

we calculate

$$\int_t^{t+r} y(t')\,dt' = \int_t^{t+r} \left\|\nabla u(t')\right\|^2 dt' \leqslant \frac{r}{\nu^2\lambda_1}\|f\|^2 + \frac{2\rho'^2}{\nu} =: k_1,$$

$$\int_t^{t+r} g(t')\,dt' = \int_t^{t+r} \frac{C(\alpha)}{\nu}\left\|u(t')\right\|^2 dt' \leqslant \frac{C(\alpha)}{\nu}\rho'^2 r =: k,$$

$$\int_t^{t+r} h(t')\,dt' = \int_t^{t+r} \left\|\nabla u(t')\right\|^2 dt' \leqslant \frac{2}{\nu}\|f\|^2 r =: k_3,$$

and thus

$$\left\|\nabla u(t)\right\|^2 \leqslant \left\{\frac{1}{\nu^2\lambda_1}\|f\|^2 + \frac{2\rho'^2}{\nu r} + \frac{C(\alpha)}{\nu}\rho'^2 r\right\} e^{\frac{2}{\nu}\|f\|^2 r} =: R_1. \tag{3.12}$$

Thus for $t \geqslant T + r$, $u(t)$ lies in a ball of radius R_1 in $\mathbf{V}$. Since R_1 is independent of u_0, (3.12) implies that for any bounded set $B \subset \mathbf{V}$,

$$\bigcup_{t \geqslant T+r} S(t)B$$

is a bounded set in $\mathbf{V}$. By the Reillich lemma (e.g., [11]), this set is compact in $\mathbf{H}$ and so $S(t)$ is uniformly compact. This completes the proof of existence of an absorbing set and compactness of $S(t)$ and thus of a global attractor.

4. Smoothness of the attractor

This section proves regularity of the maximal attractor by showing that an attractor of the NS-$\overline{\omega}$ model exists in each space $\mathbf{H}^s_{\#}$. As in [34], this implies that $\mathcal{A}$ consists of $\mathbf{C}^\infty_{\#}$ functions. So as to not overburden the presentation with further assumptions on the deconvolution operator, we consider the base model (1.1). The results in this section extend to the examples of deconvolution operators given above and to general deconvolution operators under mild assumptions on smoothing in $\mathbf{H}^s_{\#}(\Omega)$, such as

$$\left\| D(\bar{v})\right\|_{\mathbf{H}^{s+2}_{\#}} \leqslant C\|v\|_{\mathbf{H}^s_{\#}}. \tag{4.1}$$

To prove regularity of the attractor we use a bootstrap argument to prove existence for each s, sharpening the regularity proof in Theorem 3.4 in [21]. We first prove uniform boundedness of spacial derivatives.

Lemma 4.1. *Consider the NS-$\overline{\omega}$ model with $u_0 \in \mathbf{H}^s_{\#}$, $f \in \mathbf{H}^s_{\#}$, $s \geqslant 0$. Then there is a finite constant C such that*

$$\sup_{[0,\infty)} \left\| u(t)\right\|_s \leqslant \rho_s < \infty.$$

Further, for every s there is a finite constant $C(\|\nabla^s f\|, \nu, \alpha, s)$ such that

$$\int_t^{t+r} \left\|\nabla \partial^s u(t')\right\|^2 dt' \leqslant 2rC\left(\left\|\nabla^s f\right\|, \nu, \alpha, s\right) + \frac{2\rho_s^2}{\nu} < \infty. \tag{4.2}$$

Proof. We have proven this for $m = 0, 1$ in the previous section. Thus we consider $m \geqslant 2$. Letting ∂^m denote any partial derivative of order m, take an $m+1$st derivative ∂^{m+1} of the NS-$\overline{\omega}$ model. This gives

$$\left(\partial^{m+1}u\right)_t - \partial^{m+1}(u\times\overline{\omega}) - \nu\Delta\partial^{m+1}u + \nabla\partial^{m+1}P = \partial^{m+1}f, \qquad \nabla\cdot\partial^{m+1}u = 0 \tag{4.3}$$

with periodic and zero mean boundary conditions. Multiplying by $\partial^{m+1}u$, integrating and using basic inequalities gives

$$\frac{1}{2}\frac{d}{dt}\left\|\partial^{m+1}u\right\|^2 + \frac{\nu}{2}\left\|\nabla\partial^{m+1}u\right\|^2 \leqslant C\left\|\nabla^m f\right\|^2 + \left(\partial^{m+1}(u\times\overline{\omega}), \partial^{m+1}u\right), \tag{4.4}$$

where the last term is the critical one. Expanding gives

$$\begin{aligned}
\left(\partial^{m+1}(u\times\overline{\omega}), \partial^{m+1}u\right) &= \sum_{|\beta|\leqslant m+1}\binom{m+1}{\beta}\int_\Omega \partial^\beta\overline{\omega}\times\partial^{m+1-\beta}u\cdot\partial^{m+1}u\,dx \\
&= \int_\Omega \left\{\partial^{m+1}\overline{\omega}\times u\cdot\partial^{m+1}u + (m+1)\partial^m\overline{\omega}\times\partial^1 u\cdot\partial^{m+1}u + \cdots\right. \\
&\quad \left. + (m+1)\partial^1\overline{\omega}\times\partial^m u\cdot\partial^{m+1}u + \overline{\omega}\times\partial^{m+1}u\cdot\partial^{m+1}u\right\} dx.
\end{aligned}$$

By the smoothing property of the filter we have for $0 \leqslant \theta \leqslant \frac{1}{2}$ that

$$\begin{aligned}
\left(\partial^{m+1}(u\times\overline{\omega}), \partial^{m+1}u\right) &\leqslant C(m)\sum_{|\beta|\leqslant m+1}\int_\Omega \left|\partial^\beta\overline{\omega}\times\partial^{m+1-\beta}u\cdot\partial^{m+1}u\right| dx \\
&\leqslant C(m)\sum_{|\beta|\leqslant m+1}\left\|\partial^\beta\overline{\omega}\right\|_\theta\left\|\partial^{m+1-\beta}u\right\|_{\frac{1}{2}-\theta}\|u\|_{m+2} \\
&\leqslant \frac{\nu}{8}\|u\|^2_{m+2} + C(m,\nu)\sum_{|\beta|\leqslant m+1}\left\|\partial^\beta\overline{\omega}\right\|^2_\theta\left\|\partial^{m+1-\beta}u\right\|^2_{\frac{1}{2}-\theta} \\
&\leqslant \frac{\nu}{8}\|u\|^2_{m+2} + C(m,\nu,\alpha)\sum_{|\beta|\leqslant m+1}\|u\|^2_{\theta+|\beta|+1-2}\|u\|^2_{m+\frac{3}{2}-\beta-\theta}.
\end{aligned}$$

We thus have for t large enough and $0 \leqslant \theta \leqslant \frac{1}{2}$

$$\frac{1}{2}\frac{d}{dt}\left\|\partial^{m+1}u\right\|^2 + \frac{7\nu}{8}\left\|\nabla\partial^{m+1}u\right\|^2 \leqslant C\left\|\nabla^m f\right\|^2 + C(m,\nu,\alpha)\sum_{|\beta|\leqslant m+1}\|u\|^2_{|\beta|+\theta-1}\|u\|^2_{(m+1-|\beta|)+(\frac{1}{2}-\theta)}.$$

The result will be proved using the induction hypothesis provided for each β, $0 \leqslant |\beta| \leqslant m+1$, we can pick θ, $0 \leqslant \theta \leqslant \frac{1}{2}$, with

$$|\beta| + \theta - 1 \leqslant m, \quad \text{and} \quad \bigl(m + 1 - |\beta|\bigr) + \left(\frac{1}{2} - \theta\right) \leqslant m.$$

(As then each term in the above sum is uniformly bounded in t for t large enough.) From the first inequality we pick

$$\theta = 0 \quad \text{if } |\beta| = m+1 \quad \text{and} \quad \theta = \frac{1}{2} \quad \text{if } |\beta| \leqslant m.$$

If $\theta = 0$, $|\beta| = m+1$ then the second inequality becomes

$$1 + \frac{1}{2} \leqslant m + 1$$

which holds since $m \geqslant 1$. If $\theta = \frac{1}{2}$, $|\beta| \leqslant m$ the second constraint becomes

$$\bigl(m + 1 - |\beta|\bigr) \leqslant m$$

which also holds because $m \geqslant 1$. This shows that

$$\frac{d}{dt}\bigl\|\partial^{m+1}u\bigr\|^2 + \nu\bigl\|\nabla\partial^{m+1}u\bigr\|^2 \leqslant C\bigl(\bigl\|\nabla^m f\bigr\|, \nu, \alpha, m\bigr)\ (< \infty), \tag{4.5}$$

from which the result follows using the Poincaré–Friedrichs inequality and an integrating factor. The second claim follows by integrating (4.5) over $(t, t+r)$ and using standard inequalities gives

$$\bigl\|\partial^{m+1}u(t+r)\bigr\|^2 - \bigl\|\partial^{m+1}u(t)\bigr\|^2 + \int_t^{t+r} \nu\bigl\|\nabla\partial^{m+1}u\bigl(t'\bigr)\bigr\|^2\,dt' \leqslant 2rC\bigl(\bigl\|\nabla^m f\bigr\|, \nu, \alpha, m\bigr).$$

For $\|\partial^{m+1}u(t)\|$ and $\|\partial^{m+1}u(t+r)\| \leqslant \rho_{m+1}$ we have

$$\int_t^{t+r} \bigl\|\nabla\partial^{m+1}u\bigl(t'\bigr)\bigr\|^2\,dt' \leqslant 2rC\bigl(\bigl\|\nabla^m f\bigr\|, \nu, \alpha, m\bigr) + \frac{2\rho_{m+1}^2}{\nu} < \infty, \tag{4.6}$$

completing the proof. □

The above lemma gives the necessary a priori bounds to apply the uniform Gronwall lemma as in the $s = 0$ case. Thus we can conclude existence of an attractor.

Theorem 4.2 *(Smooth attractors). Suppose*

$$u_0, f \in \mathbf{H}^s_\#(\Omega), \quad \nabla \cdot u_0 = \nabla \cdot f = 0, \qquad \int_\Omega f\,dx = \int_\Omega u_0\,dx = 0.$$

Then the NS-$\overline{\omega}$ model (1.1) *has a global attractor in each $\mathbf{H}^s_\#(\Omega)$ and thus consists of $\mathbf{C}^\infty_\#(\Omega)$ functions.*

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